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Dispersion and Dispersivity Tensors in Saturated Porous Media with Uniaxial Symmetry

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Abstract The coefficient of dispersion, D_{ij}, and the dispersivity, a_{ijkl}, appear in the expression for the flux of a solute in saturated flow through porous media. We present a detailed analysis of these tensors in an axially symmetric porous medium, e.g., a stratified porous medium, with alternating layers, and show that in such a medium, the dispersivity is governed by 6 independent moduli. We present also the constraints that have to be satisfied by these moduli. We also show that at least two independent experiments are required in order to obtain the values of these coefficients for any three-dimensional porous medium domain.

Keywords Transport in porous media, solute transport, · anisotropy · dispersion coefficient · dispersivity and axially symmetric porous medium

1 Introduction

The coefficients of dispersion, D_{ij}, appears in the Fickian-type expression for the dispersive flux of a solute (mass of solute per unit area of void space in a porous medium cross-section) in saturated flow through porous media (Bear, 1972),

$$J_i = -\sum_{j=1}^{3} D_{ij} \nabla_j c, \quad D_{ij} = D_{ji}, \quad i, j = 1, 2, 3 \tag{1}$$

where J_i denotes the ith component of the solute's 3-dim flux vector, $\mathbf{J}$, and c is the solute's average concentration. Equation (1) is valid for the general case of any anisotropic porous medium, with isotropic medium as a special case.

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In contrast to similar linear flux laws, e.g., Ohm's and Fourier's laws, with corresponding tensorial coefficients of electroconductivity and of thermoconductivity, respectively, which depend only on properties of the considered medium, here, in the case of the dispersive flux in saturated flow, the dispersion tensor D_{ij} depends also on the fluid's average velocity $\mathbf{U}$ in the void space (Bear, 1961), (Scheidegger, 1961),

$$D_{ij} = \frac{1}{U} \sum_{k,l=1}^{3} a_{ijkl} \, U_k U_l, \qquad \mathbf{U}^2 = \sum_{k=1}^{3} U_k^2, \qquad \mathbf{U}^2 = U^2, \tag{2}$$

where a_{ijkl} is a property, called *dispersivity*, of the porous medium only, and U_i is the ith component of the fluid's average velocity vector $\mathbf{U}$.

Since the early 60's, almost all research on (solute) dispersion has been limited to isotropic porous media. For such media, the components of the dispersivity tensor have been shown to depend only on two material moduli, referred to as *longitudinal* and *transversal* dispersivities (Bear, 1961, Nikolaevskii, 1959, Scheidegger, 1961). Based on the works of Robertson (1940) and Batchelor (1946), another study of the dispersivity of axially symmetric porous media was done by Poreh (1965), leading to the conclusion that four material moduli are required to describe all a_{ijkl}-components. In the last four decades, to our best knowledge, no theoretical work based on both properties of the inner symmetry of the tensor-a_{ijkl}, i.e. permutation indices rules, and external symmetry of porous media, i.e. point symmetry groups, has been undertaken.

A comprehensive review of the dispersion phenomenon in axisymmetric porous media is presented by Lichtner *et al*, 2002. In the practice of ground water hydrology, on the basis of field observations, it has been suggested that for uniform flow parallel to the horizontal stratification in a layered aquifer, transverse dispersion is much smaller in the vertical direction than in the horizontal one (see Garabedian *et al*, 1991, Gelhar *et al*, 1992, Robson, 1978). Based on such observations, a 'working model' for axially symmetric porous media was suggested by Burnett and Frind (1987), Jensen *et al*, 1993, and Zheng and Bennett, 1995, in which the dispersion tensor was defined by *three* dispersivities only: a longitudinal dispersivity, in the direction of stratification, and two transversal dispersivities, a horizontal one and a vertical one. Moreover, no theoretical proof has ever been provided, and no attempt has ever been made to analyze the dispersivity tensor a_{ijkl} for anisotropic porous media, with various non-uniaxial symmetries.

The objective in this paper is to derive an explicit expression for the dispersivity tensor, focusing on axially symmetric media. We also determine the constraints that have to be satisfied by the dispersivity moduli and discuss the number of independent experiments which are required to obtain the values of these moduli for any porous medium domain.

2 Symmetry considerations

The dispersivity is a 4th rank tensor, with intrinsic symmetry of permutation indices $\left[V^2\right]^2$ in Jahn's notations (Jahn, 1949),

$$\left[V^2\right]^2 \; : \; a_{ijkl} = a_{ijlk} \, , \quad a_{ijkl} = a_{jikl}. \tag{3}$$

In dealing with the dispersivity tensor, additional simplifications can be obtained by making use of certain auxiliary algebraic relations among its components. These relations appear under the action of the crystallographic point symmetry group $\mathcal{G} \subset \mathcal{O}(3)$ of the porous medium that makes some (not all) of the 36 elements of a_{ijkl} survive, or at least independent of the remaining ones. According to Sirotine and Chaskolskaya (1984), the decrease in the number $N(\mathcal{G})$ of a_{ijkl}-components is significant when symmetry is increased, i.e.,

$$N(\mathcal{C}_1) = 36 \;\rightarrow\; \ldots \;\rightarrow\; N(\mathcal{D}_{2h}) = 12 \;\rightarrow\; N(\mathcal{D}_{4h}) = 7 \;\rightarrow\; N(\mathcal{D}_{\infty h}) = 6 \;\rightarrow\; N(\mathcal{O}(3)) = 2 \tag{4}$$

In thermodynamics, the rate of entropy production, σ, is related to the thermodynamic driving force, $\mathbf{X}$, and to the thermodynamic flux, $\mathbf{Y}$. According to De Groot and Mazur (1962), two vectors $\mathbf{X}$ and $\mathbf{Y}$ are considered conjugated if they obey the relationship

$$\sigma = \sum_{i=1}^{3} Y_i X_i. \tag{5}$$

In the case considered here, the dispersive flux of the solute, $\mathbf{J}$, is driven by ∇c, which acts as a 'driving force.' In other words, $\mathbf{X} = g\,\nabla c$ and $\mathbf{Y} = g\,\mathbf{J}$, where g denotes a (dimensional) scalar parameter, which depends on the considered transport phenomenon. In this case, the rate of entropy production is expressed by

$$\sigma = g^2 \sum_{i,j=1}^{3} D_{ij}\ \nabla_i c\ \nabla_j c. \tag{6}$$

By requiring that the quadratic form (6) be positive definite, we derive the following constraints on the D_{ij}-matrix,

$$D_{ii} > 0,\ \ \det \begin{Bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{Bmatrix} > 0,\ \ \det \begin{Bmatrix} D_{11} & D_{13} \\ D_{31} & D_{33} \end{Bmatrix} > 0,\ \ \det \begin{Bmatrix} D_{22} & D_{23} \\ D_{23} & D_{33} \end{Bmatrix} > 0,\ \ \det D_{ij} > 0.$$

The above constraints hold irrespective of the fluid's averaged velocity $\mathbf{U}$. Furthermore, by (2), they provide another set of constraints imposed on elements of the dispersivity tensor, a_{ijkl}. Such straightforward way to establish these constraints is not effective, since it requires a lot of routine algebra and does not prevent repetitive inequalities. In section 5, we shall present a much more effective method for deriving the sought constraints.

In the meantime, let us estimate the total number of such inequalities. Making use of the 2-dim matrix representation of the a_{ijkl}-tensor (Sirotine and Chaskolskaya, 1984), one can show that the number $M(\mathcal{G})$ of such constraints decreases also with growth of the symmetry $\mathcal{G}$ of the porous medium,

$$M(\mathcal{C}_1) = 63 \;\rightarrow\; \ldots \;\rightarrow\; M(\mathcal{D}_{2h}) = 10 \;\rightarrow\; M(\mathcal{D}_{4h}) = 7 \;\rightarrow\; M(\mathcal{D}_{\infty h}) = 6 \;\rightarrow\; M(\mathcal{O}(3)) = 2. \tag{7}$$

3 The Tensor a_{ijkl} in a Porous Medium with Uniaxial Symmetry

In this section, we study the 3-dim tensor a_{ijkl}, whose components depend only on porous medium properties, actually, for saturated flow, only on the geometry of the medium's void space. We shall focus on porous media with uniaxial symmetry. When averaged, a stratified horizontal aquifer with alternating, relatively thin layers, is an example of such medium. Our interest in this kind of porous medium is motivated by two reasons. First, in saturated flow through aquifers, the porous medium comprising the latter is often axially symmetric (de Marsily, 86), with a vertical axis, e.g., in a layered aquifer. On the other hand, four decades of continuing discussions (Lichtner *et. al.*, 2002) and references therein) on explicit expressions for the tensors a_{ijkl} and D_{ij} in porous media with uniaxial symmetry have not provided any rigorous analysis of these entities.

Instead of a 2-dim matrix representation of the a_{ijkl}-tensor, presented by Sirotine and Chaskolskaya (1984), we suggest another representation, which originates in the theory of uniaxial nematic liquid crystals (LC). The underlying idea is that when we compare the a_{ijkl}-tensor with the 4th rank viscosity tensor, η_{ijkl}, of a uniaxial LC, its intrinsic symmetry of permutation indices is $\left[\left[V^2\right]^2\right]$,

$$\left[\left[V^2\right]^2\right] \quad : \quad \eta_{ijkl} = \eta_{ijlk}, \quad \eta_{ijkl} = \eta_{jikl}, \quad \eta_{ijkl} = \eta_{klij}. \tag{8}$$

i.e. slightly 'stronger' than (3). According to Landau and Lifshitz (1986), the known representation of η_{ijkl} is

$$\begin{aligned}\eta_{ijkl} &= \eta_1 \delta_{ij}\delta_{kl} + \eta_2 \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \eta_{34} \left(e_i e_j \delta_{kl} + e_k e_l \delta_{ij}\right) \\ &\quad + \eta_5 \left(e_i e_k \delta_{jl} + e_j e_k \delta_{il} + e_i e_l \delta_{jk} + e_j e_l \delta_{ik}\right) + \eta_6 e_i e_j e_k e_l,\end{aligned} \tag{9}$$

in which the e_i's denote the three components of the unit vector $\mathbf{e}$, along the symmetry axis of the medium.

From (9), it follows that there exists only one term, $(e_i e_j \delta_{kl} + e_k e_l \delta_{ij})$, which reduces its intrinsic symmetry: $\left[\left[V^2\right]^2\right] \rightarrow \left[V^2\right]^2$, after decomposition of this term into two separate terms: $e_i e_j \delta_{kl}$ and $e_k e_l \delta_{ij}$. This explains why, in accordance with (7), the tensor a_{ijkl} has six independent moduli and how to construct its most generic form:

$$\begin{aligned}a_{ijkl} &= a_1\ \delta_{ij}\delta_{kl} + \frac{a_2}{2} \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + a_3\ e_i e_j \delta_{kl} + a_4\ e_k e_l \delta_{ij} + \\ &\quad \frac{a_5}{2} \left(e_i e_k \delta_{jl} + e_j e_k \delta_{il} + e_i e_l \delta_{jk} + e_j e_l \delta_{ik}\right) + a_6\ e_i e_j e_k e_l.\end{aligned} \tag{10}$$

Note that by making $e_i \rightarrow 0$ in (10), we obtain the dispersivity tensor for an isotropic porous medium,

$$a_{ijkl} = a_1 \delta_{ij}\delta_{kl} + \frac{a_2}{2} \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{11}$$

The axisymmetric tensor (10) has to be supplemented by a set of constraints that result from the thermodynamic requirement imposed on the rate of entropy production, σ, defined in (6).

4 The Tensor D_{ij} in a Porous Medium with Uniaxial Symmetry

In this section, we consider the 3-dim tensor D_{ij} and certain interesting related phenomena. Based on (2) and (10), we rewrite this tensor as

$$UD_{ij} = a_1\mathbf{U}^2\delta_{ij} + a_2U_iU_j + a_3\mathbf{U}^2e_ie_j + a_4\langle\mathbf{e},\mathbf{U}\rangle^2\delta_{ij} + a_5\langle\mathbf{e},\mathbf{U}\rangle(U_ie_j + e_iU_j) + a_6\langle\mathbf{e},\mathbf{U}\rangle^2e_ie_j \quad (12)$$

where $\langle\mathbf{e},\mathbf{U}\rangle$ denotes a scalar product of the two vectors: $\mathbf{e}$ and $\mathbf{U}$.

It is interesting to note that Batchelor (1946), in his work on axisymmetric turbulence, derived a general expression for D_{ij} that is based on five tensorial terms: e_ie_j, δ_{ij}, U_iU_j, U_ie_j, U_je_i, and four scalar functions, $C_i(\mathbf{U},\mathbf{e})$,

$$UD_{ij} = C_1(\mathbf{U},\mathbf{e})e_ie_j + C_2(\mathbf{U},\mathbf{e})\delta_{ij} + C_3(\mathbf{U},\mathbf{e})U_iU_j + C_4(\mathbf{U},\mathbf{e})(U_ie_j + U_je_i). \quad (13)$$

Keeping in mind that there exist only two quadratic-in-U scalar invariants, U^2 and $\langle\mathbf{U},\mathbf{e}\rangle^2$, and only one linear-in-U scalar invariant, $\langle\mathbf{U},\mathbf{e}\rangle$, we can verify the equivalence of the two expressions, (12) and (13), by noting that

$$C_1(\mathbf{U},\mathbf{e}) = a_3U^2 + a_6\langle\mathbf{U},\mathbf{e}\rangle^2,\ C_2(\mathbf{U},\mathbf{e}) = a_1U^2 + a_4\langle\mathbf{U},\mathbf{e}\rangle^2,\ C_3(\mathbf{U},\mathbf{e}) = a_2,\ C_4(\mathbf{U},\mathbf{e}) = a_5\langle\mathbf{U},\mathbf{e}\rangle.$$

We introduce the two 6-dim vectors, $\mathbf{D}_6$ and $\mathbf{a}_6$,

$$\mathbf{D}_6 = \{D_{11}, D_{22}, D_{33}, D_{12}, D_{23}, D_{31}\}, \quad \mathbf{a}_6 = \{a_1, a_2, a_3, a_4, a_5, a_6\}, \quad (14)$$

and denote their transposed counterparts by $\mathbf{D}_6^T$ and $\mathbf{a}_6^T$. Next, we rewrite (12) in the form

$$\mathbf{D}_6^T = \frac{1}{U}\,\varDelta_1(\mathbf{U},\mathbf{e})\cdot\mathbf{a}_6^T, \quad (15)$$

where the 6×6 matrix $\varDelta_1(\mathbf{U},\mathbf{e})$ is defined as

$$\varDelta_1(\mathbf{U},\mathbf{e}) = \left\{\begin{array}{cccccc} \mathbf{U}^2 & U_1^2 & e_1^2\mathbf{U}^2 & \langle\mathbf{e},\mathbf{U}\rangle^2 & 2\langle\mathbf{e},\mathbf{U}\rangle e_1U_1 & \langle\mathbf{e},\mathbf{U}\rangle^2e_1^2 \\ \mathbf{U}^2 & U_2^2 & e_2^2\mathbf{U}^2 & \langle\mathbf{e},\mathbf{U}\rangle^2 & 2\langle\mathbf{e},\mathbf{U}\rangle e_2U_2 & \langle\mathbf{e},\mathbf{U}\rangle^2e_2^2 \\ \mathbf{U}^2 & U_3^2 & e_3^2\mathbf{U}^2 & \langle\mathbf{e},\mathbf{U}\rangle^2 & 2\langle\mathbf{e},\mathbf{U}\rangle e_3U_3 & \langle\mathbf{e},\mathbf{U}\rangle^2e_3^2 \\ 0 & U_1U_2 & e_1e_2\mathbf{U}^2 & 0 & \langle\mathbf{e},\mathbf{U}\rangle(U_1e_2+e_1U_2) & \langle\mathbf{e},\mathbf{U}\rangle^2e_1e_2 \\ 0 & U_2U_3 & e_2e_3\mathbf{U}^2 & 0 & \langle\mathbf{e},\mathbf{U}\rangle(U_2e_3+e_2U_3) & \langle\mathbf{e},\mathbf{U}\rangle^2e_2e_3 \\ 0 & U_3U_1 & e_3e_1\mathbf{U}^2 & 0 & \langle\mathbf{e},\mathbf{U}\rangle(U_3e_1+e_3U_1) & \langle\mathbf{e},\mathbf{U}\rangle^2e_3e_1 \end{array}\right\}. \quad (16)$$

Equation (15) opens the way to the derivation of the 6 dispersivity a_i-moduli from values of D_{ij}. The latter can be obtained from experimental measurements, assuming that the symmetry axis $\mathbf{e}$ is known, and so is the velocity $\mathbf{U}$, in an appropriate reference frame, x_1, x_2, x_3. However, a straightforward calculation shows that the matrix $\varDelta_1(\mathbf{U},\mathbf{e})$ is degenerate,

$$\det\varDelta_1(\mathbf{U},\mathbf{e}) = 0, \quad \mathrm{rank}\,\varDelta_1(\mathbf{U},\mathbf{e}) = 4\,. \quad (17)$$

Equality (17) leads to two most important consequences and raises a question:

1. At most, four independent components D_{ij} can be obtained from a single experiment. The remaining two D_{ij}'s are linearly representable by the first four.
2. One cannot get all six dispersivity a_i-moduli from a single experiment.
3. How many experiments do we have to perform in order to obtain all six a_i-values?

In what follows, we shall demonstrate that two different experiments that suffice for obtaining all six a_i-values. We consider two cases (= experiments) in a homogeneous, axially symmetric domain:

1. $U_1 = 0$, $U_2 = U_3 = u$ and $e_1 = 1$, $e_2 = e_3 = 0$. Then,

$$D_{11} = 2(a_1 + a_3)u, \quad D_{22} = D_{33} = (2a_1 + a_2)u, \quad D_{23} = a_2 u, \quad D_{12} = D_{31} = 0 \quad (18)$$

The above relationships provide three moduli: a_1, a_2 and a_3.

2. $U_1 = U_2 = u$, $U_3 = 0$ and $e_1 = 1$, $e_2 = e_3 = 0$. Then,

$$\begin{aligned} D_{11} &= (2a_1 + a_2 + 2a_3 + a_4 + 2a_5 + a_6)u, \quad D_{12} = (a_2 + a_5)u, \\ D_{22} &= (2a_1 + a_2 + a_4)u, \quad D_{33} = (2a_1 + a_4)u, \quad D_{23} = D_{31} = 0\ . \end{aligned} \quad (19)$$

The above relations provide three additional moduli: a_4, a_5 and a_6.

Thus, by two experiments, with different setups of $\mathbf{U}$, and a known $\mathbf{e}$, we have obtained the entire set of 6 dispersivity moduli in a 3-dim saturated porous medium with uniaxial symmetry.

4.1 The Tensor D_{ij} in an Isotropic Porous Medium

For an isotropic porous medium, instead of (12), we have

$$U D_{ij} = a_1 U^2 \delta_{ij} + a_2 U_i U_j. \quad (20)$$

This means that only two moduli: a_1 and a_2 are required for a full description of the D_{ij}-components. The expression for $\Delta_1\,(\mathbf{U}, \mathbf{e})$-matrix, which is defined in (16), is even more degenerated, with $rank \Delta_1\,(\mathbf{U}, \mathbf{e}) = 2$. This indicates a strong linear dependence among the D_{ij}-values: there exist four linear relations among the six D_{ij}-elements.

Let us consider an isotropic porous medium, with the setup: $U_1 = u$, $U_2 = U_3 = 0$. Then,

$$((D_{ij})) = u \left\{ \begin{array}{ccc} a_1 + a_2 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{array} \right\}. \quad (21)$$

In ground water hydrology (Bear, 1972), the modulus a_1 is called transversal dispersivity (denoted a_T) and $a_1 + a_2$ is called longitudinal dispersivity (denoted a_L).

5 Thermodynamic constraints

In this section we determine the algebraic constraints that have to be satisfied by the six dispersivity a_i-moduli for an axisymmetric porous medium. We substitute (12) into (6) and present σ in its invariant form, irrespective of the reference frame,

$$\begin{aligned} U \frac{\sigma}{g^2} = a_1\, \mathbf{U}^2\, (\nabla c)^2 + a_2\, \langle \mathbf{U}, \nabla c \rangle^2 + a_3\, \mathbf{U}^2 \langle \mathbf{e}, \nabla c \rangle^2 + a_4\, \langle \mathbf{e}, \mathbf{U} \rangle^2\, (\nabla c)^2 \\ +2\, a_5\, \langle \mathbf{e}, \mathbf{U} \rangle \langle \mathbf{U}, \nabla c \rangle \langle \nabla c, \mathbf{e} \rangle + a_6\, \langle \mathbf{e}, \mathbf{U} \rangle^2 \langle \mathbf{e}, \nabla c \rangle^2\ . \end{aligned} \quad (22)$$

We then represent (22) as a quadratic form. This is a standard way to write the necessary and sufficient conditions for σ to be positive definite. For this purpose, we choose the reference frame such that $e_1 = e_2 = 0$ and $e_3 = 1$. By substituting these into (22), we get

$$U\frac{\sigma}{g^2} = a_1 \sum_{i=1}^{3} U_i^2 \sum_{i=1}^{3} (\nabla_i c)^2 + a_2 \left(\sum_{i=1}^{3} U_i \nabla_i c\right)^2 + a_3 (\nabla_3 c)^2 \sum_{i=1}^{3} U_i^2 + a_4 U_3^2 \sum_{i=1}^{3} (\nabla_i c)^2$$
$$+2\, a_5 U_3 \nabla_3 c \sum_{i=1}^{3} U_i \nabla_i c + a_6 U_3^2 (\nabla_3 c)^2 \ . \tag{23}$$

Expression (23) is a quadratic form in a 9-dim space, with orthogonal basis $U_i \nabla_j c$, $i, j = 1, 2, 3$. We decompose it in the form

$$U\frac{\sigma}{g^2} = a_1 \left[U_1^2 (\nabla_2 c)^2 + U_2^2 (\nabla_1 c)^2\right] + (a_1 + a_3) (\nabla_3 c)^2 \left[U_1^2 + U_2^2\right] + (a_1 + a_4) U_3^2 \left[(\nabla_1 c)^2 + (\nabla_2 c)^2\right]$$
$$+ (a_1 + a_2) \left[U_1^2 (\nabla_1 c)^2 + U_2^2 (\nabla_2 c)^2\right] + (a_1 + a_2 + a_3 + a_4 + 2a_5 + a_6) U_3^2 (\nabla_3 c)^2$$
$$+ 2a_2 U_1 U_2 \nabla_1 c \nabla_2 c + 2(a_2 + a_5) U_3 \nabla_3 c \left(U_1 \nabla_1 c + U_2 \nabla_2 c\right) \ , \tag{24}$$

and require the positive definiteness of the last expression. Then we obtain three inequalities

$$a_1 > 0 \ , \quad a_1 + a_3 > 0 \ , \quad a_1 + a_4 > 0 \ , \tag{25}$$

and requirement of positive definiteness of the 3×3 matrix Γ,

$$\Gamma = \left\{ \begin{array}{ccc} a_1 + a_2 & a_2 & a_2 + a_5 \\ a_2 & a_1 + a_2 & a_2 + a_5 \\ a_2 + a_5 & a_2 + a_5 & a_1 + a_2 + a_3 + a_4 + 2a_5 + a_6 \end{array} \right\} . \tag{26}$$

The last claim leads to three additional algebraically independent restrictions,

$$a_1 + a_2 > 0 \ , \quad a_1 + a_2 + a_3 + a_4 + 2a_5 + a_6 > 0 \ , \tag{27}$$
$$a_1^2 + a_1(3a_2 + a_3 + a_4 + 2a_5 + a_6) + 2a_2(a_3 + a_4 + a_6) > 2a_5^2 \ .$$

Thus, we have arrived at 6 constraints imposed on the 6 dispersivity a_i-moduli; this is in full agreement with (7) for uniaxial symmetry group $\mathcal{G} = \mathcal{D}_{\infty h}$. Note that the first two inequalities in (25) and (27) correspond to isotropic porous media.

6 The Tensor D_{ij} in a Planar Porous Medium with Dihedral Symmetry

In this section we consider the dispersion phenomenon in a 2-dim porous medium domain, which is obtained by intersecting a 3-dim uniaxial porous medium with a plane that is parallel to the uniaxial axis. In the resulting 2-dim porous medium domain, there are two mirror lines, which are perpendicular to each other. By these symmetry elements, it is easy to recognize the dihedral point symmetry group $\mathcal{D}_2$, which is one of ten 2-dim point symmetry subgroups of $\mathcal{O}(2)$.

From the mathematical standpoint, the dispersion tensor, D_{ij}, can be represented by a 2×2 symmetric matrix, with three components: $\mathbf{D}_3 = \{D_{11}, D_{22}, D_{12}\}$, instead of the 6×6 matrix $\Delta_2(\mathbf{U}, \mathbf{e})$, as in (15). By doing so, we obtain:

$$\mathbf{D}_3^T = \frac{1}{U}\, \Delta_2(\mathbf{U}, \mathbf{e}) \cdot \mathbf{a}_6^T , \tag{28}$$

where the 3×6 matrix, $\Delta_2(\mathbf{U}, \mathbf{e})$, is defined as

$$\Delta_2(\mathbf{U}, \mathbf{e}) = \left\{ \begin{array}{cccccc} \mathbf{U}^2 & U_1^2 & e_1^2\mathbf{U}^2 & \langle \mathbf{e}, \mathbf{U}\rangle^2 & 2\langle \mathbf{e}, \mathbf{U}\rangle e_1 U_1 & \langle \mathbf{e}, \mathbf{U}\rangle^2 e_1^2 \\ \mathbf{U}^2 & U_2^2 & e_2^2\mathbf{U}^2 & \langle \mathbf{e}, \mathbf{U}\rangle^2 & 2\langle \mathbf{e}, \mathbf{U}\rangle e_2 U_2 & \langle \mathbf{e}, \mathbf{U}\rangle^2 e_2^2 \\ 0 & U_1U_2 & e_1e_2\mathbf{U}^2 & 0 & \langle \mathbf{e}, \mathbf{U}\rangle(U_1e_2 + e_1U_2) & \langle \mathbf{e}, \mathbf{U}\rangle^2 e_1e_2 \end{array} \right\}. \quad (29)$$

Equation (28) does not enable the determination of the six dispersivity a_i-moduli, from 3 values of D_{ij}, which can be obtained from a single experiment. For comparison, in section 4, we have shown that by two experiments, with different setups of $\mathbf{U}$, and a known $\mathbf{e}$, we have obtained the entire set of 6 dispersivity moduli in a 3-dim saturated porous medium with uniaxial symmetry.

In this conjunction, the following question arises: how many experiments do we have to perform in a 2-dim saturated porous medium domain with dihedral symmetry, in order to obtain the entire set of 6 dispersivity moduli?

It is easy to show that two experiments are not enough. Indeed, let us consider two different flow setups, with fluid averaged velocities $\mathbf{W}$ and $\mathbf{U}$, and the same orientation of $\mathbf{e}$,

$$_1\mathbf{D}_3^T = \frac{1}{U}\, \Delta_2(\mathbf{W}, \mathbf{e}) \cdot \mathbf{a}_6^T\ , \qquad _2\mathbf{D}_3^T = \frac{1}{U}\, \Delta_2(\mathbf{U}, \mathbf{e}) \cdot \mathbf{a}_6^T. \quad (30)$$

In (30) the front subscripts 1 and 2 in the notations $_1\mathbf{D}_3^T$ and $_2\mathbf{D}_3^T$, indicate data obtained from the 1st and 2nd experiments, with velocities $\mathbf{W}$ and $\mathbf{U}$, respectively. We now introduce a new 6-dim vector, $\mathsf{D}_{6,a}$,

$$\mathsf{D}_{6,a} = \{_1D_{11},\ _1D_{22},\ _1D_{12},\ _2D_{11},\ _2D_{22},\ _2D_{12}\}, \quad (31)$$

composed of two 3-dim vectors $_1\mathbf{D}_3 = \{_1D_{11},\ _1D_{22},\ _1D_{12}\}$ and $_2\mathbf{D}_3 = \{_2D_{11},\ _2D_{22},\ _2D_{12}\}$, and rewrite the two equations in (30), in the form

$$\mathsf{D}_{6,a}^T = \frac{1}{U}\, \Delta_3(\mathbf{W}, \mathbf{U}, \mathbf{e}) \cdot \mathbf{a}_6^T\ , \quad \text{where} \quad (32)$$

$$\Delta_3(\mathbf{W}, \mathbf{U}, \mathbf{e}) = \left\{ \begin{array}{cccccc} \mathbf{W}^2 & W_1^2 & e_1^2\mathbf{W}^2 & \langle \mathbf{e}, \mathbf{W}\rangle^2 & 2\langle \mathbf{e}, \mathbf{W}\rangle e_1 W_1 & \langle \mathbf{e}, \mathbf{W}\rangle^2 e_1^2 \\ \mathbf{W}^2 & W_2^2 & e_2^2\mathbf{W}^2 & \langle \mathbf{e}, \mathbf{W}\rangle^2 & 2\langle \mathbf{e}, \mathbf{W}\rangle e_2 W_2 & \langle \mathbf{e}, \mathbf{W}\rangle^2 e_2^2 \\ 0 & W_1W_2 & e_1e_2\mathbf{W}^2 & 0 & \langle \mathbf{e}, \mathbf{W}\rangle(W_1e_2 + e_1W_2) & \langle \mathbf{e}, \mathbf{W}\rangle^2 e_1e_2 \\ \mathbf{U}^2 & U_1^2 & e_1^2\mathbf{U}^2 & \langle \mathbf{e}, \mathbf{U}\rangle^2 & 2\langle \mathbf{e}, \mathbf{U}\rangle e_1 U_1 & \langle \mathbf{e}, \mathbf{U}\rangle^2 e_1^2 \\ \mathbf{U}^2 & U_2^2 & e_2^2\mathbf{U}^2 & \langle \mathbf{e}, \mathbf{U}\rangle^2 & 2\langle \mathbf{e}, \mathbf{U}\rangle e_2 U_2 & \langle \mathbf{e}, \mathbf{U}\rangle^2 e_2^2 \\ 0 & U_1U_2 & e_1e_2\mathbf{U}^2 & 0 & \langle \mathbf{e}, \mathbf{U}\rangle(U_1e_2 + e_1U_2) & \langle \mathbf{e}, \mathbf{U}\rangle^2 e_1e_2 \end{array} \right\}.$$

A straightforward calculation shows that

$$\det \Delta_3(\mathbf{W}, \mathbf{U}, \mathbf{e}) = 0, \quad \operatorname{rank} \Delta_3(\mathbf{W}, \mathbf{U}, \mathbf{e}) = 5. \quad (33)$$

Thus, among the six elements of the vector $\mathsf{D}_{6,a}$, given in (31), there are only five independent ones, and equation (32) does not suffice for providing the six dispersivity a_i-moduli.

We shall now show that three experiments are sufficient for providing the entire set of a_i's. Consider 3 different flow setups, with velocities: $\mathbf{W}$, $\mathbf{U}$ and $\boldsymbol{\Upsilon}$ and with the same orientation of $\mathbf{e}$,

$$_1\mathbf{D}_3^T = \frac{1}{U}\, \Delta_2(\mathbf{W}, \mathbf{e}) \cdot \mathbf{a}_6^T\ , \quad _2\mathbf{D}_3^T = \frac{1}{U}\, \Delta_2(\mathbf{U}, \mathbf{e}) \cdot \mathbf{a}_6^T\ , \ , \quad _3\mathbf{D}_3^T = \frac{1}{U}\, \Delta_2(\boldsymbol{\Upsilon}, \mathbf{e}) \cdot \mathbf{a}_6^T, \quad (34)$$

where ${}_1\mathbf{D}_3$ and ${}_2\mathbf{D}_3$ were defined earlier and ${}_3\mathbf{D}_3 = \{{}_3D_{11},\ {}_3D_{22},\ {}_3D_{12}\}$. In contrast to (31), we introduce another 6-dim vector, $\mathsf{D}_{6,b}$,

$$\mathsf{D}_{6,b} = \{{}_1D_{11},\ {}_1D_{12},\ {}_2D_{11},\ {}_2D_{12},\ {}_3D_{11},\ {}_3D_{12}\}, \tag{35}$$

composed of two elements, ${}_iD_{11}$ and ${}_iD_{12}$, of three 3-dim vectors ${}_i\mathbf{D}_3$, $i = 1, 2, 3$. We rewrite the three equations (34) in the form,

$$\mathsf{D}^T_{6,b} = \frac{1}{U}\ \Delta_4\left(\mathbf{W}, \mathbf{U}, \boldsymbol{\Upsilon}, \mathbf{e}\right) \cdot \mathbf{a}^T_6\ , \quad \text{where} \tag{36}$$

$$\Delta_4\left(\mathbf{W}, \mathbf{U}, \boldsymbol{\Upsilon}, \mathbf{e}\right) = \begin{Bmatrix} \mathbf{W}^2 & W_1^2 & e_1^2\mathbf{W}^2 & \langle\mathbf{e}, \mathbf{W}\rangle^2 & 2\langle\mathbf{e}, \mathbf{W}\rangle e_1 W_1 & \langle\mathbf{e}, \mathbf{W}\rangle^2 e_1^2 \\ 0 & W_1W_2 & e_1e_2\mathbf{W}^2 & 0 & \langle\mathbf{e}, \mathbf{W}\rangle(W_1e_2 + e_1W_2) & \langle\mathbf{e}, \mathbf{W}\rangle^2 e_1e_2 \\ \mathbf{U}^2 & U_1^2 & e_1^2\mathbf{U}^2 & \langle\mathbf{e}, \mathbf{U}\rangle^2 & 2\langle\mathbf{e}, \mathbf{U}\rangle e_1 U_1 & \langle\mathbf{e}, \mathbf{U}\rangle^2 e_1^2 \\ 0 & U_1U_2 & e_1e_2\mathbf{U}^2 & 0 & \langle\mathbf{e}, \mathbf{U}\rangle(U_1e_2 + e_1U_2) & \langle\mathbf{e}, \mathbf{U}\rangle^2 e_1e_2 \\ \boldsymbol{\Upsilon}^2 & \Upsilon_1^2 & e_1^2\boldsymbol{\Upsilon}^2 & \langle\mathbf{e}, \boldsymbol{\Upsilon}\rangle^2 & 2\langle\mathbf{e}, \boldsymbol{\Upsilon}\rangle e_1 \Upsilon_1 & \langle\mathbf{e}, \boldsymbol{\Upsilon}\rangle^2 e_1^2 \\ 0 & \Upsilon_1\Upsilon_2 & e_1e_2\boldsymbol{\Upsilon}^2 & 0 & \langle\mathbf{e}, \boldsymbol{\Upsilon}\rangle(\Upsilon_1e_2 + e_1\Upsilon_2) & \langle\mathbf{e}, \boldsymbol{\Upsilon}\rangle^2 e_1e_2 \end{Bmatrix}.$$

Note that the three 2-dim vectors: $\mathbf{W}$, $\mathbf{U}$ and $\boldsymbol{\Upsilon}$ are always coplanar. However, if at least two of them would be collinear, then (36) cannot be solved uniquely, i.e. $\det \Delta_4\left(k\mathbf{U}, \mathbf{U}, \boldsymbol{\Upsilon}, \mathbf{e}\right) = 0$, where k is a real number. Otherwise, a straightforward calculation shows that if neither two of the three vectors: $\mathbf{W}$, $\mathbf{U}$, and $\boldsymbol{\Upsilon}$ are collinear, then

$$\det \Delta_4\left(\mathbf{W}, \mathbf{U}, \boldsymbol{\Upsilon}, \mathbf{e}\right) \neq 0, \quad \operatorname{rank} \Delta_4\left(\mathbf{W}, \mathbf{U}, \boldsymbol{\Upsilon}, \mathbf{e}\right) = 6. \tag{37}$$

Thus, three special (not any) experiments suffice for providing all six dispersivity a_i-moduli in a 2-dim saturated porous medium with dihedral symmetry.

7 Concluding Remarks

For the case of an axisymmetric porous medium, we have shown that six independent a_i-moduli are needed in order to determine all components of the dispersivity tensor, and the constraints that these moduli have to satisfy. This information is required when determining the latter by experiments for specific porous media. We have also found the number of experiments that are required to determine the entire set of dispersivity moduli for 2-dim and 3-dim domains. This is important for experimental determination of the dispersivity.

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