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RESEARCH ARTICLE

The Global Approximated Particular solution Meshless Method for Two-dimensional Linear Elasticity problems

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The two-dimensional linear elasticity equations are solved by the Global Method of Approximate Particular Solutions (MAPS) as a new meshless option to the conventional Finite Element discretization. The displacement components are approximated by a linear combination of the elasticity particular solutions and the stress tensor is obtained by differentiating the displacement expressions in terms of the particular solutions. The Multiquadric Radial Basis Function (RBF) is employed as the non-homogeneous term in the governing equation to compute the particular solutions. The cantilever beam and the infinite plate with a hole problems are solved to verify the implemented meshless method. For each situation the trend of the root mean square error is assessed in terms of the shape parameter and the number of nodes. Unlike most of the RBF collocation strategies, it is found that numerical results are in good agreement to the analytical solutions for a wide range of shape parameter values.

Keywords: Particular solutions; Plane elasticity; Radial Basis Functions; Meshless methods

AMS Subject Classification: 65N80;74S30;G.1.8

1. Introduction

Recently several meshless methods has been implemented for solving elasticity problems as an option to overcome the limitations found with the Finite Element (FEM) and the Finite Difference (FDM) methods in situations which involve large deformation processes, mesh enrichment or moving boundaries [1, 22, 28]. The element-free Galerkin method (EFGM) proposed by Belytschko et al. [5] has become the meshless method most widely used in solid mechanics [29]. In the EFGM the weak form of the governing equations is solved in the sense of the FEM using a background cells for integration and shape functions derived from a moving least squares (MLS) scheme. With the aim of reducing computational effort and improving accuracy of the EFGM several methods has been implemented such as the Reproducing Kernel Particle Meshless method (RKPM) [18], the partition of unity method (PUM) [4], the point interpolation method (PIM) [17] and the meshless local Petrov Galerkin (MLPG) approach [2].

However most of the weak form meshless methods are not truly meshless since they require a grid to perform integration and, in consequence, their computational

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efficiency is affected [23]. Therefore strong form meshless methods arise as a suitable strategy to solve elasticity problems by using only nodes distributed along the domain and its boundaries. Collocation strategies are the most popular option among the strong form methods since they can overcome the problems found in Galerkin methods due to the use of approximation instead of interpolations [18]. In this sense, Zhang et al. [28] employed Radial Basis Function (RBFs) collocation to solve the Poisson and elasticity equations highlighting some advantages such as the truly meshless character of the method, dimensional independency and spectral convergence order of some RBFs, showing that the RBF collocation approach is a suitable meshless alternative to FEM in elasticity problems. Nevertheless globally supported RBFs produce ill-conditioned interpolation matrices.

Although the ill-conditioning of the interpolation matrix and the sensitivity of the numerical solutions respect to the shape parameter value have limited the development of the global RBF collocation methods, Kansa's [14] pioneering global approach (direct method) has been successfully used for the solution of several boundary value problems governing by different PDEs, see [14], Laplace, [8, 14, 20], Poisson, [8], Helmholtz, and [6, 8], convection diffusion, among others.

Robert Schaback [25] described the behaviour of the RBF direct collocation as the uncertainty relation: Better conditioning is associated with worse accuracy, and worse conditioning is associated with improved accuracy. As the system size is increased, this problem becomes more pronounced. Several strategies have been developed to overcome the matrix ill-conditioning of the global RBF collocation methods; among the easier of implementation the following two approaches are worthy to mention: the use of RBF-specific matrix preconditioners [7], and the adaptive selection of functional centres and collocation points [15].

Despite of the improvements made, the accuracy of the direct RBF global collocation methods is still limited by the matrix ill conditioning. New alternatives based on the use of different RBFs schemes have been recently proposed. The indirect approach approximates the higher derivatives appearing in the PDE to be solved by a RBF interpolation and the expression for the dependent variable is obtained by integrating them with respect to the corresponding coordinates (see Mai-Duy and Tran-CongMay [19, 21]). By using the indirect strategy, it is possible to obtain accurate solutions for a wider range of shape parameter than those obtained with the direct approach, where the dependent variable is approximated by a RBF interpolation and its derivatives obtained by the differentiation of the interpolation function. According to Mai-Duy and Tran-CongMay, this is due to the smooth behaviour of the integration process, which unlike derivation does not contain inherent inaccuracy of the approximation.

Recently, Chen and Fan [9] proposed an integrated RBF method for the case of a linear PDE, which differential operator $L(.(\vec{x}))$, or only part of it, can be written in terms of the radial component of a polar or spherical coordinate systems, as:

$$L(u(\vec{x})) = L1_r(u(r)) + L2_{\vec{x}}(u(\vec{x})) = f(\vec{x}), \quad (1)$$

In this case, these authors proposed to approximate the radial component of the PDE, i.e. $L1_r(.(r))$, in terms of a RBF interpolation and by integrating the resulting non-homogeneous ordinary differential equation, it is possible to obtain an approximated representation of the field variable by a superposition of the corresponding particular solution. This is similar to the integrated RBF approach (Mai-Duy's et al. [19] indirect scheme), but instead of approximating the higher order derivatives in terms of a RBF and integrating them in each direction along a

Cartesian grid to obtain an approximation of the field variable, the approximated field variable is found in term of the complete integration of the operator $L1_r$ independently of any Cartesian grid. In this way, the following approximation is defined:

$$L1_r(u(r)) = \sum_{k=1}^{N} \alpha_k \phi(r_k), \tag{2}$$

with corresponding field variable as:

$$u(\vec{x}) = \sum_{k=1}^{N} \alpha_k \hat{u}(r_k) \tag{3}$$

where the particular solution $\hat{u}(r)$ is given by solving (4), in terms of the RBF nonhomogeneous term $\phi(r)$,

$$L1_r(\hat{u}(r)) = \phi(r) \tag{4}$$

By substituting the obtained approximation of $u(\vec{x})$ into the full expression of $L(u(\vec{x}))$ and into the boundary conditions of the problem, taking into consideration (2), a meshless integrated RBF solution of the boundary value problem is obtained, which is free of the definition of a direct or auxiliary Cartesian grid. This integrated RBF meshless approach has been referred by Chen and Fan [9] as the Method of Approximate Particular Solution (MAPS).

In the case of a linear boundary value problem,

$$L(u(\vec{x})) = f(\vec{x}) \quad \forall \vec{x} \in \Omega \tag{5}$$

and

$$B(u(\vec{x})) = g(\vec{x}) \quad \forall \vec{x} \in \Gamma \tag{6}$$

with PDE operator $L(u)$ defined only in terms of the radial coordinate, i.e. $L(u) = L1_r(u)$, and the boundary operator given by B. The implementation of the MAPS reduces to the solution of the following linear system of algebraic equations:

$$\begin{pmatrix} B[\hat{u}(\vec{x}_1, \vec{\xi}_1)] & \cdots & B[\hat{u}(\vec{x}_1, \vec{\xi}_N)] \\ \vdots & \ddots & \vdots \\ B[\hat{u}(\vec{x}_{N_b}, \vec{\xi}_1)] & \cdots & B[\hat{u}(\vec{x}_{N_b}, \vec{\xi}_N)] \\ \phi(|\vec{x}_{N_b+1} - \vec{\xi}_1|) & \cdots & \phi(|\vec{x}_{N_b+1} - \vec{\xi}_N|) \\ \vdots & \ddots & \vdots \\ \phi(|\vec{x}_N - \vec{\xi}_1|) & \cdots & \phi(|\vec{x}_N - \vec{\xi}_N|) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{N_b} \\ \alpha_{N_b+1} \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} g(\vec{x}_1) \\ \vdots \\ g(\vec{x}_{N_b}) \\ f(\vec{x}_{N_b+1}) \\ \vdots \\ f(\vec{x}_N) \end{pmatrix} \tag{7}$$

for N_b boundary points and N_i internal points, with $N = N_b + N_i$ (see Figure 1). The solution of the above boundary value problem is achieved after solving the resulting algebraic system for the coefficients α.

The particular solution $\hat{u}$ of (4), has been used before in connection with the numerical solution of a linear boundary value problem using the classical decomposition in term of its particular and homogeneous solutions, combined with the Method of Fundamental solution (see [11]) and with the Boundary Element method (see [10]) to find the corresponding homogeneous solution.

The main difficulty of the MAPS is the possibility of obtaining a close form of the particular solution, which in some cases can be found with the help of symbolic computation. In this paper the two dimensional elasticity equations are solved by means of the global MAPS, using a well-known procedure to obtain the elasticity particular solutions as is shown in [24]. The proposed MAPS scheme is validated by comparing the numerical results with the analytical solution of two different boundary value problems, for several nodal distributions and shape parameter values.

2. The particular solution method for Linear Elasticity problems

The momentum conservation equation for elastic solids, neglecting gravitational effects, is given by the following expression

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \tag{8}$$

where σ_{ij} is the ij-component of the stress tensor, which is defined in terms of the directional derivatives of the displacement components, u_i, as:

$$\sigma_{ij} = \frac{\bar{E}}{2(1+\bar{\nu})}\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{2\bar{\nu}}{1-2\bar{\nu}}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]. \tag{9}$$

The constants $\bar{E}$ and $\bar{\nu}$ are functions of the actual elasticity young modulus E and the Poisson's constant ν and their values depend on the applied assumption; for plane strain

$$\bar{E} = E, \tag{10}$$

$$\bar{\nu} = \nu, \tag{11}$$

and for plane stress

$$\bar{E} = \frac{E(1+2\nu)}{(1+\nu)^2}, \tag{12}$$

$$\bar{\nu} = \frac{\nu}{1+\nu}. \tag{13}$$

After substituting expression (9) into the momentum conservation equation (8), the following equation in terms of displacements is obtained (Navier's equations)

$$\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{1-2\bar{\nu}} \frac{\partial^2 u_j}{\partial x_i \partial x_j} = 0. \quad (14)$$

The boundary value problem to be solved is given by equation (14), with essential (Dirichlet), given displacements, and natural (Neumann), given surface tractions, boundary conditions

$$u_i = u_{ib} \quad \forall \vec{x} \in \Gamma_u, \quad (15)$$

$$t_i = \sigma_{ij} n_j = t_{ib} \quad \forall \vec{x} \in \Gamma_t. \quad (16)$$

Let consider the following inhomogeneous Navier's equations:

$$\frac{\partial^2 u_i^l}{\partial x_j \partial x_j} + \frac{1}{1-2\bar{\nu}} \frac{\partial^2 u_j^l}{\partial x_i \partial x_j} = \frac{2(1+\bar{\nu})}{\bar{E}} \phi(r) \delta_{il}. \quad (17)$$

where u_i^l are the displacement particular solutions for the elasticity problem with the Multiquadric RBF $\phi = \sqrt{(r^2+c^2)}$ as the external force, which is only a function of the Euclidean distance $r = ||\vec{x} - \vec{\xi}||$ and the shape parameter c. Here, the shape parameter is constant and its influence on the numerical solution is assessed in the section of numerical results. The solution of equation (17) is attained by expressing the particular solution in terms of the Galerkin vector V_{ij} as

$$u_i^l(\vec{x}, \vec{\xi}) = \frac{\partial^2 V_{il}(\vec{x}, \vec{\xi})}{\partial x_j \partial x_j} - \frac{1}{2(1-\bar{\nu})} \frac{\partial^2 V_{ij}(\vec{x}, \vec{\xi})}{\partial x_l \partial x_j}. \quad (18)$$

If the Galerkin vector is write in terms of the scalar potential ψ according to

$$V_{ij} = \frac{2(1+\bar{\nu})}{\bar{E}} \psi \delta_{ij}, \quad (19)$$

and the equation (18) is substituted into the inhomogeneous Navier's equations (17), the following bi-harmonic equation for ψ is obtained

$$\frac{\partial^4 \psi(r)}{\partial x_j \partial x_j \partial x_k \partial x_k} = \phi(r). \quad (20)$$

The equation (20) is solved by direct integration using the Wolfram Mathematica Online Integrator tool. The resulting singular components are subtracted from the final equation knowing that such terms are complementary solutions of the bi-harmonic equation [24]. In the cases of two dimensional problems, as the one considered in this work, the final expression for ψ is given as:

$$\psi(r) = \frac{1}{12}\left[\frac{1}{75}(r^2+c^2)^{1/2}(4r^4+48r^2c^2-61c^4) - c^3ln(c)r^2 \right.$$
$$\left. -\frac{1}{5}(5r^2-2c^2)c^3ln(c+(r^2+c^2)^{1/2})\right]. \quad (21)$$

A similar expression to (21) is reported in [24], where the term $c^3ln(c)r^2$ is omitted, when implementing the Dual Reciprocity Boundary Element Method (BE-DRM) for two-dimensional elastodynamics problems.

The displacement particular solutions are found by substituting equation (21) into the expressions (19) and (18). The particular solutions for the stress tensor σ^l_{ij} are obtained by replacing the directional derivatives of the displacement particular solution into the stress tensor definition (9). For brevity, the resulting equations for the displacement components and the stress tensor are given in the appendix 4.

The limiting value as r tends to zero of the obtained particular solutions, displacements and stresses, are not where singular in a bounded domain Ω, for more details see the appendix 4. Therefore, the approximated displacement vector $\vec{u}$ and stress tensor $\bar{\bar{\sigma}}$ can be defined as a linear superposition of N particular solutions located at N trail points ξ_k in the form:

$$u_i(\vec{x}) = \sum_{k=1}^{N} \alpha^l_k u^l_i(\vec{x}, \vec{\xi}_k) \quad (22)$$

$$\sigma_{ij}(\vec{x}) = \sum_{k=1}^{N} \alpha^l_k \sigma^l_{ij}(\vec{x}, \vec{\xi}_k), \quad (23)$$

By substituting the above expression into the Navier's equations (14), it is found the following homogeneous linear superposing of MQ functions representing the approximation of the governing system of PDEs:

$$\sum_{k=1}^{N} \alpha^l_k \left[\frac{\partial^2 u^l_i(\vec{x}, \vec{\xi}_k)}{\partial x_j \partial x_j} + \frac{1}{1-2\bar{\nu}}\frac{\partial^2 u^l_j(\vec{x}, \vec{\xi}_k)}{\partial x_j \partial x_i}\right] = \sum_{k=1}^{N} \alpha^l_k \left[\phi(r_k)\delta_{il}\right] = 0 \quad (24)$$

where equation (17) has been taken into account.

The equation set required to complete the collocation process is obtained by substituting the approximations (22) and/or (23) into the respective boundary condition (15) and/or (16). In general, the boundary conditions, essential or natural, can be represented as $B^i(u_1, u_2) = g_i(\vec{x})$, where B^i is the boundary differential operator which applies in the i-direction. The form of B depends on the type of the boundary condition, i.e., whether is essential (15) or natural (16). Collocating the resulting expression at the N_{b1} boundary nodes on Γ_u and/or at the N_{b2} boundary nodes on Γ_t, for each the component $k = 1, 2$, the first two set of lines on the matrix system (25) are found, while collocation of (24) at the N_i internal nodes, for each the component $k = 1, 2$, defines the last two equation sets of the matrix system

$$\begin{pmatrix} B^1\left[u_1^1, u_2^1\right] & B^1\left[u_1^2, u_2^2\right] \\ B^2\left[u_1^1, u_2^1\right] & B^2\left[u_1^2, u_2^2\right] \\ [\phi] & [0] \\ [0] & [\phi] \end{pmatrix} \left(\begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix} \right) = \begin{pmatrix} [g(\vec{x})_1] \\ [g(\vec{x})_2] \\ [0] \\ [0] \end{pmatrix}. \tag{25}$$

After solving the above system of equations for $[\alpha^1]$ and $[\alpha^2]$, the displacement and stress tensor components can be obtained anywhere in the domain by evaluating expressions (22) and (23).

3. Numerical Results

Two plane elasticity problems with analytical solution are employed to validate the proposed MAPS approach. The first of them is a 2D cantilever beam which is solved for essential (first case) and mixed, i.e. combination of essential and natural, (second case) type of boundary conditions. The second problem is an infinite plate with a hole subjected to uniaxial traction. In both cases, the Root Mean Square (RMS) error defined as

$$\epsilon_\phi = \sqrt{\frac{1}{N}\sum_{i=1}^{N} |\phi_r(\vec{x}_i) - \phi(\vec{x}_i)|^2} \tag{26}$$

is employed to have a global quantitative measure of the agreement between analytical and numerical solutions. In equation (26), ϕ refers to the variable used to compute the error ($\phi_r(\vec{x}_i)$ analytical and $\phi(\vec{x}_i)$ numerical) and N to the number of evaluation points. Also, the RMS error is employed to analyse the influence of the shape parameter on the numerical solution.

3.1 *The 2-D cantilever beam*

The exact solutions of the cantilever beam with unit thickness and an applied parabolic surface traction profile at the free end (Figure 2), is given by Timoshenko and Goodier [26] as (plane stress):

$$u_1 = \frac{P}{6EI}\left(x_2 - \frac{D}{2}\right)\left[(6L - 3x_1)x_1 + (2+\nu)(x_2^2 - 2Dx_2)\right] \tag{27}$$

$$u_2 = -\frac{P}{6EI}\left[3\nu(x_2 - \frac{1}{2}D)^2(L - x_1) + \frac{1}{4}(4+5\nu)D^2x_1 + (3L - x_1)x_1^2\right] \tag{28}$$

and

$$\sigma_{11} = \frac{P}{I}(L - x_1)\left(x_2 - \frac{1}{2}D\right) \tag{29}$$

$$\sigma_{12} = \frac{Px_2}{2I}(x_2 - D) \tag{30}$$

$$\sigma_{22} = 0 \tag{31}$$

where I is the cantilever moment of inertia, $\frac{D^3}{12}$. All numerical solutions are obtained for $E = 1000$, $\nu = \frac{1}{3}$, $P = 0.2$, $L = 1.0$ and $D = 0.2$. In the first case displacement boundary conditions are fixed at all boundary points by using the analytical solution. The corresponding RMS errors are shown in Table 1 for the displacements and stresses, for a value of $c = 0.025$, which is not the optimal value but a value, among the experimentally tested, that allows to obtain accurate solutions for all the used nodal distributions. The results reported in the table are in excellent agreement with the analytical solution, even in the case of the coarsest mesh, 15×5. The corresponding relative errors to the RMS errors reported in Table 1, with respect to the maximum values of $u_{1max} = 0.015$ and $\sigma_{11max} = 30.0$, are respectively 0.01 and 0.45% for the 15×5 nodal distribution and 7.97×10^{-5}% and 8.31×10^{-3}% for 63×21 distribution. The accuracy of the present results is of the same order of magnitude or slightly higher than those obtained by Zhang et al. [28] with a global Kansa and Hermite RBF collocation schemes, and of the same order of magnitude that those found by Tolstykh and Shirobokob [27] using a local RBF generalized finite difference scheme.

In the second case, the analytical displacements are applied at the line $x_1 = 0.0$, zero traction at $x_2 = 0.0$ and $x_2 = D$ and analytical traction at $x_1 = L$. It is known that the solution of the cantilever problem with these type of mixed boundary condition is more computational demanding than the previous one with only Dirichlet conditions, as commented in [3]. The RMS errors shown in Table 2 are found with a value of $c = 0.01$ for the same nodal distributions used in the first case. Although the obtained relative error for the stress field, σ_{ij}, is unacceptable high in the case of the coarsest nodal distributions, ($\approx 15\%$), adequate solutions are attained for finer distributions such as 31×11 and 63×21, with respectively relative errors of 0.23 and 0.044% for u_1, and 2.15 and 0.047% for σ_{11}. Our results show higher order of accuracy than the weighed least square approximation of the Kansa's global collocation method reported by Hu et al. [12], with almost twice the number of equations than unknowns, but lower than the corresponding solutions of the same approach with scaled boundary conditions, also reported by Hu et al. [12]. Besides, the accuracy of our results are several order of magnitude higher than those obtained with a RBF compact support collocation reported by Kangzu et al. [13].

The agreement between the analytical solution and the numerical result can be verified by looking at the line contours of constant shear rate σ_{11} presented in Figure 3, where the numerical result was obtained with a 63×21 uniform nodal distribution and a value of $c = 0.01$. A relative small difference between the contours can be observed at the region $0.0 < x_1 < 0.1$, towards the upper and lower boundaries, which is due to the use of a homogeneous nodal distribution even in regions with high field gradients. These small differences can be improved by refining the nodal distribution towards those regions as shown in [1].

The convergence of the method as the number of points increases, for each of the problem considered and values of the shape parameter used, can be also observed in tables 1 and 2, respectively. On the other hand, the influence of the shape parameter on the accuracy of the solution is shown in Figure 4 for a nodal distribution of 51×17 points, where the variation of the longitudinal displacement (u_1) absolute RMS error in terms of the shape parameter for the two cases considered is reported. In both cases, the RMS error is almost constant up to a value of $c \approx 1 \times 10^{-3}$ within the range of values of c evaluated, with the difference that in the first case (Dirichlet boundary conditions) the error is smaller than in the second case (mixed boundary conditions). Besides, in the second case, a local minimum of the RMS error is found at the value of $c = 0.01$, which is only slightly smaller than the constant value found for smaller values of c.

From the obtained results, it can be concluded that in the present case of a 2-D cantilever problem, the proposed MAPS is stable for a larger range of values of the shape parameter. Producing accurate solutions, with relative errors less than 0.7% in a 51×17 nodal distribution, when $1 \times 10^{-5} < c < 2 \times 10^{-2}$. For shape parameters greater than 0.02, the resulting global matrix becomes too ill-conditioned to be solved with the direct algorithm used in this work.

In comparison with other global RBF meshless collocation approaches, which solutions are in general very sensitive to the value of the shape parameter, the proposed MAPS produces consistent accurate solutions for a large range of the value of the shape parameter, even when using very small value of c. It is important to point out that in the present formulation it is always necessary to have a value of c different from zero in order to have a regular solution, since the obtained expressions for the displacements and stresses particular solutions are singular as r tends to zero when $c = 0$. In the present numerical example a value of c as smaller as 10^{-5} is enough to find an accurate solution. For values of c smaller than 10^{-5}, it is not possible to obtain accurate solution.

The present results show that in the case of a cantilever problem, the proposed global MAPS can achieve high order of accuracy in the prediction of the displacement and stress fields, even that is well known that the use of global collocation methods in slender domains leads to inaccuracy in the computation of the stress field, as commented in [16].

3.2 *Infinite plate with a hole*

Consider the infinite plate with a circular hole at the centre subjected to a uniaxial traction P as is shown in Figure 5, which exact analytical solution for the stress and displacement fields is given by Timoshenko and Goodier [26] in terms of the polar coordinates r and θ as:

$$\sigma_{11}(r,\theta) = P\left\{1 - \frac{a^2}{r^2}\left[\frac{3}{2}cos(2\theta) + cos(4\theta)\right] + \frac{3a^4}{2r^4}cos(4\theta)\right\} \tag{32}$$

$$\sigma_{22}(r,\theta) = -P\left\{\frac{a^2}{r^2}\left[\frac{1}{2}cos(2\theta) - cos(4\theta)\right] + \frac{3a^4}{2r^4}cos(4\theta)\right\} \tag{33}$$

$$\sigma_{12}(r,\theta) = -P\left\{\frac{a^2}{r^2}\left[\frac{1}{2}sin(2\theta) + sin(4\theta)\right] - \frac{3a^4}{2r^4}sin(4\theta)\right\}, \tag{34}$$

and

$$u_1(r,\theta) = \frac{P}{8G}\left\{\frac{r}{a}(\kappa+1)cos\theta + \frac{2a}{r}\left[(1+\kappa)cos\theta + cos(3\theta)\right] - \frac{2a^3}{r^3}cos(3\theta)\right\} \quad (35)$$

$$u_2(r,\theta) = \frac{P}{8G}\left\{\frac{r}{a}(\kappa-3)sin\theta + \frac{2a}{r}\left[(1-\kappa)sin\theta + sin(3\theta)\right] - \frac{2a^3}{r^3}sin(3\theta)\right\}, \quad (36)$$

where a is the hole radius, $G = \frac{E}{2(1+\nu)}$ is the shear modulus and $\kappa = 3 - 4\nu$ is the Kolosov constant for the plain strain assumption. For the present numerical solution, the problem domain is defined by the zone bounded by dotted line in Figure 5 (left), which is shown in the right hand side of the same figure. The domain length L along the x_1 and x_2 directions is set equal to $5a$ and the boundary conditions are defined by the given analytical solution, with given displacements on lines $x_1 = 0.0$ and $x_2 = 0.0$ and given tractions on lines $x_1 = L$ and $x_2 = L$. A free traction boundary condition is applied on the hole edge $x_1^2 + x_2^2 = a^2$.

The RMS errors presented in Table 3 correspond to the numerical solution attained with $\nu = 0.3$, $E = 1000$ and $a = 1$ by using the nodal distributions shown in Figure 6 (a-d). The nodal distribution shown in Figure b. is a globally refined version of the distribution in a., the distribution presented in c. is the configuration b. refined towards the hole border, and the distribution d. is b. refined near the hole border and the domain corners.

Unlike in the cantilever beam problem, in the present case different shape parameter values were used in order to obtain well-conditioned matrices for the nodal distributions given in Figure 6 (a-d). Enough accurate solution is obtained even when using the coarsest nodal distribution of 225 points, the RMS errors in Table 3 for $N = 225$ correspond to relative errors of 0.12, 0.17, 0.51,0.92 and 0.87% for u_1, u_2, σ_{11}, σ_{12} and σ_{22}, respectively.

Excellent agreement between the analytical and the numerical results is observed in the Figure 7, where the numerical stress contours has been obtained with the finest nodal distribution (872 nodes). As in the previous case of the cantilever problem, the accuracy of the present results is of the same order of magnitude or slightly higher than previously reported global and even local RBFs meshless collocation solutions (see Zhang et al. [28], Tolstykh and Shirobokob [27], Hu et al. [12], and Kangzu et al. [13]).

Besides, in our results almost the same accuracy is obtained for values of $1 \times 10^{-4} < c < 1 \times 10^{-1}$, when employing the 477-node distribution, as is presented in Figure 8. Showing that as before, in this case, the proposed global MAPS is stable for a wide range of shape parameters values.

4. Conclusions

For the first time, the global MAPS has been employed to solve two-dimensional linear Elasticity problems. The developed meshless numerical scheme is defined in terms of linear superposition of particular solutions of the inhomogeneous Navier system of equations with the Multiquadric RBF as the source term. The proposed MAPS is verified by computing the relative RMS error in the numerical solution of two classical linear elasticity problems with known analytical solution; the 2-D cantilever beam and the infinite plate with a hole. In the case of the infinite plate with a hole, it was necessary to refine the nodal distribution collocation points towards the hole edge and the domain corners to be able to obtain a high resolution solution.

The obtained numerical results are stable and accurate for a wide range of shape parameter values and several nodal distributions of collocation points. Unlike most of the other global RBF collocation schemes, in the present case, the RMS error remain almost constant as the shape parameter value tends to zero, however if the value of c is too small no accurate solution can be found. On the other hand, like most of the global RBF collocation approaches, the shape parameter values are restricted by the conditioning of the global matrix which worsens with the increase of the c value. In the present formulation it is always necessary to have a value of c different from zero, since the obtained expressions for the displacements and stresses particular solutions are singular as r tends to zero when $c = 0$. The reported numerical results show the capability of the proposed Global MAPS to solve two-dimensional linear Elasticity problems, with high order of accuracy in the numerical results for a large range of the value of the shape parameter used.

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Appendix A. Displacement particular solutions

The displacement particular solutions, in terms of the potential ψ is given by:

$$u_i^l(\vec{x},\vec{\xi}) = \frac{2(1+\bar{\nu})}{\bar{E}}\left\{\left(\frac{\psi'}{r}+\psi''\right)\delta_{il} - \frac{1}{2(1-\bar{\nu})}\left[\frac{\psi'}{r}\left(\delta_{il}-\frac{\hat{x}_i\hat{x}_j}{r^2}\right)+\psi''\frac{\hat{x}_i\hat{x}_j}{r^2}\right]\right\} \quad (37)$$

where $\hat{x}_i = x_i - \xi_i$ $(i = 1, 2)$, with corresponding stress particular solutions

$$\sigma_{ij}^l(\vec{x},\vec{\xi}) = \left[\frac{\bar{\nu}}{1-\bar{\nu}}\left(\frac{\psi'}{r^2}-\frac{\psi''}{r}\right)+\psi'''\right]\left(\frac{\hat{x}_j}{r}\delta_{il}+\frac{\hat{x}_i}{r}\delta_{jl}\right) + \left[\left(\frac{\psi'}{r^2}-\frac{\psi''}{r}\right)+\frac{\bar{\nu}}{1-\bar{\nu}}\psi'''\right]\frac{\hat{x}_l}{r}\delta_{ij} - \left[3\left(\frac{\psi'}{r^2}-\frac{\psi''}{r}\right)+\psi'''\right]\frac{\hat{x}_i\hat{x}_j\hat{x}_l}{(1-\bar{\nu})r^3} \quad (38)$$

where

$$\frac{\psi'}{r} = \frac{4r^6 + 36r^4c^2 + 39r^2c^4 + 7c^6}{180\left(r^2+c^2\right)^{(3/2)}} - \frac{\left(5r^4 + 3r^2c^2 - 2c^4\right)c^3}{60\left(r^2+c^2\right)^{(3/2)}\left[c + \left(r^2+c^2\right)^{(1/2)}\right]} - \frac{1}{6}c^3 ln\left[c + \left(r^2+c^2\right)^{(1/2)}\right] - \frac{1}{6}c^3 ln\left(c\right) \quad (39)$$

$$\psi'' = \frac{16r^6 + 84r^4c^2 + 96r^2c^4 + 7c^6}{180\left(r^2+c^2\right)^{(3/2)}} - \frac{\left(20r^4 + 25r^2c^2 - 2c^4\right)c^3}{60\left(r^2+c^2\right)^{(3/2)}\left[c + \left(r^2+c^2\right)^{(1/2)}\right]} + \frac{\left(5r^2 - 2c^2\right)c^3r^2}{60\left(r^2+c^2\right)\left[c + \left(r^2+c^2\right)^{(1/2)}\right]^2} - \frac{1}{6}c^3 ln\left[c + \left(r^2+c^2\right)^{(1/2)}\right] - \frac{1}{6}c^3 ln\left(c\right) \quad (40)$$

$$\psi''' = \frac{\left(76r^4 + 176r^2c^2 + 285c^4\right)r}{300\left(r^2+c^2\right)^{(3/2)}} + \frac{\left(4r^4 + 48r^2c^2 - 61c^4\right)r^3}{300\left(r^2+c^2\right)^{(5/2)}} + \frac{\left(10r^4 + 15r^2c^2 - 2c^4\right)c^3r}{20\left(r^2+c^2\right)^2\left[c + \left(r^2+c^2\right)^{(1/2)}\right]} - \frac{\left(5r^2 + 22c^2\right)c^3r}{20\left(r^2+c^2\right)^{(3/2)}\left[c + \left(r^2+c^2\right)^{(1/2)}\right]} + \frac{\left(-5r^2 + 2c^2\right)c^3r^3}{20\left(r^2+c^2\right)^{(5/2)}\left[c + \left(r^2+c^2\right)^{(1/2)}\right]} + \frac{\left(-5r^2 + 2c^2\right)c^3r^3}{30\left(r^2+c^2\right)^{(3/2)}\left[c + \left(r^2+c^2\right)^{(1/2)}\right]^3} \quad (41)$$

As can be observed, the above expressions are non-singular in any bounded domain. The singularities as r tends to zero of $\frac{\psi'}{r^2}$ and $\frac{\psi''}{r}$ are cancelled in the expression $\frac{\psi'}{r^2} - \frac{\psi''}{r}$, appearing in (38).

References

[1] M.H. Afshar, J. Amani, and M. Naisipour, *A node enrichment adpatative refinement in discrete least squares meshless mehtod for solution of elasticity problems*, Eng. Anal. Bound. Elem. 36 (2012), pp. 385–393.
[2] S.N. Alturi and T. Zhu, *A new meshless local petrov galerkin (mlpg) approach in computational mechanics*, Comput. Mech. 22 (1998), pp. 223–235.
[3] C.E. Augarde and A.J. Deeks, *The use of timoshenko's exact solution for a cantilever beam in adaptive analysis*, Finite Elements in Analysis and Design 44 (2008), pp. 595–601.
[4] I. Babuska and J.M. Melenk, *The partition of unity method*, Int. J. for Numer. Methods Engrg. 40 (1997), pp. 727–740.
[5] T. Belytschko, Y.Y. Lu, and L. Gu, *Element-free galerkin methods*, Int. J. for Numer. Methods Engrg. 37 (1994), pp. 229–256.
[6] I. Boztosuna and A. Charafi, *An analysis of the linear advection-diffusion equation using mesh-free and mesh-dependent methods*, Eng. Anal. Bound. Elem. 26 (2002), pp. 889–895.
[7] D. Brown, *On approximate cardinal preconditioning methods for solving pdes with radial basis functions*, Eng. Anal. Bound. Elem. 29 (2005), pp. 343–353.
[8] S. Chantasiriwan, *Cartesian grid methods using radial basis functions for solving poisson, helmholtz, and diffusion convection equations*, Eng. Anal. Bound. Elem. 28 (2004), pp. 1417–1425.
[9] C. Chen, C. Fan, and P. Wen, *The method of approximated particular solutions for solving certain partial differential equations*, Numer. Meth. Part. D. E. (2010), p. doi:10.1002/num.20631.
[10] W.F. Florez and H. Power, *Drm multidomain mass conservative interpolation approach for the bem solution of the two-dimensional navier-stokes equations*, Comput. Math. Appl. 43 (2002), pp. 457–472.

[11] M. Golberg and C. Cheng, *The method of fundamental solutions for potential, Helmholtz and diffusion problems*, in *Boundary Integral Methods: Numerical and Mathematical Aspects*, Comput. Mech. Publ., Southampton, 1998, pp. 103–176.

[12] H.Y. Hu, J.S. Chen, and W. Hu, *Weighted radial basis collocation method for boundary value problems*, International Journal for Numerical Methods in Engineering 69 (2007), pp. 2736–2757.

[13] S. Kangzu, Z. Xiong, and L. Mingwan, *Meshless method based on collocation with consistent compactly supported radial basis functions*, ACTA Mechanica Sinica 20 (2004), pp. 551–557.

[14] E.J. Kansa, *Multiquadrics -a scattered data approximation scheme with applications to computational fluid dynamics-ii solution to parabolic, hyperbolic and elliptic partial differential equations*, Comput. Math. Appl. 19 (1990), pp. 127–145.

[15] L. Ling, R. Opfer, and R. Schaback, *Results on meshless collocation techniques*, Eng. Anal. Bound. Elem. 30 (2006), pp. 247–253.

[16] T. Liszka, C. Duarte, and W. Tworzydlo, *hp-meshless cloud method*, Comput. Methods Appl. Mech. Engrg. 139 (1996), pp. 263–288.

[17] G.R. Liu and Y.T. Gu, *A point interpolation method for two-dimensional solids*, Int. J. for Numer. Methods Engrg. 50 (2001), pp. 937–951.

[18] W.K. Liu, S. Jun, and Y.F. Zhang, *Reproducing kernel particle methods*, Int. J. Numer. Methods Fluids 20 (1995), pp. 1081–1106.

[19] N. Mai-Duy and T. Tran-Cong, *Numerical solution of diferential equations using multiquadric radial basis function networks*, Neural Networks 14 (2001), pp. 185–199.

[20] ———, *Mesh-free radial basis function network methods with domain decomposition for approximation of functions and numerical solution of poisson's equations*, Eng. Anal. Bound. Elem. 26 (2002), pp. 133–156.

[21] ———, *Approximation of function and its derivatives using radial basis function networks*, Appl. Math. Model. 27 (2003), pp. 197–220.

[22] W. Quak, A.H. van den Boogard, D. González, and E. Cueto, *A comparative study on the performance of meshless approximations and their integration*, Comput. Mech. 48 (2011), pp. 121–137.

[23] A. Sadeghirad, I.M. Kani, M. Rahimian, and A.V. Astaneh, *Meshfree local integration on line method (mlilm) for linear elasticity*, The Arabian Journal for Science and Engineering 33 (2008), pp. 411–434.

[24] M.F. Samaan and Y.F. Rashed, *Free vibration multiquadric boundary elements applied to plane elasticity*, Appl. Math. Model. 33 (2009), pp. 2421–2432.

[25] R. Schaback, *Multivariate interpolation and approximation by translates of basis functions*, in *Approximation Theory VIII: Wavelets and Multilevel Approximation*, World Scientific Pub Co Inc, College Station, 1995, pp. 1–8.

[26] S. Timoshenko and J.N. Goodier, *Theory of Elasticity*, Mc Graw-Hill, New York (1951).

[27] A.I. Tolstykh and D. Shirobokov, *On using radial basis functions in a "finite difference mode" with application to elasticity problems*, Computational Mechanics 33 (2003), pp. 68–79.

[28] X. Zhang, Z. Song, M.W. Lu, and X. Liu, *Meshless methods based on collocation with radial basis functions*, Comput. Mech. 26 (2000), pp. 333–343.

[29] X. Zhuang, C. Heaney, and C. Auguarde, *On error control in the element-free galerkin method*, Eng. Anal. Bound. Elem. 36 (2012), pp. 351–360.

N	ϵ_{u_1}	ϵ_{u_2}	$\epsilon_{\sigma_{11}}$	$\epsilon_{\sigma_{12}}$	$\epsilon_{\sigma_{22}}$
15×5	1.3149×10^{-5}	1.0465×10^{-5}	1.1697×10^{0}	6.2343×10^{-1}	5.5946×10^{-1}
21×7	5.9931×10^{-6}	4.0686×10^{-6}	6.3254×10^{-1}	3.3043×10^{-1}	2.8383×10^{-1}
31×11	1.9988×10^{-6}	1.3017×10^{-6}	3.0233×10^{-1}	1.4829×10^{-1}	1.2309×10^{-1}
45×15	8.4133×10^{-7}	5.5424×10^{-7}	1.4947×10^{-1}	7.5349×10^{-2}	6.3345×10^{-2}
51×17	6.1077×10^{-7}	4.1012×10^{-7}	1.1662×10^{-1}	5.8047×10^{-2}	5.3614×10^{-2}
63×21	4.3477×10^{-7}	2.7276×10^{-7}	9.0670×10^{-2}	7.2926×10^{-2}	4.4604×10^{-2}

Table 1. Absolute RMS error for the solution of the 2D cantilever beam problem (First case)

N	ϵ_{u_1}	ϵ_{u_2}	$\epsilon_{\sigma_{11}}$	$\epsilon_{\sigma_{12}}$	$\epsilon_{\sigma_{22}}$
15×5	4.5440×10^{-3}	2.5581×10^{-2}	$3.9737 \times 10^{+1}$	4.0233×10^{0}	5.4189×10^{0}
21×7	3.9651×10^{-3}	1.6117×10^{-2}	$1.6279 \times 10^{+1}$	2.7850×10^{0}	4.8621×10^{0}
31×11	6.4618×10^{-4}	6.2315×10^{-3}	$1.1931 \times 10^{+1}$	2.4055×10^{0}	5.0529×10^{0}
45×15	4.3648×10^{-4}	2.0674×10^{-3}	1.2454×10^{0}	3.0548×10^{-1}	3.4378×10^{-1}
51×17	3.1637×10^{-4}	1.6566×10^{-3}	8.0172×10^{-1}	2.0269×10^{-1}	2.2369×10^{-1}
63×21	2.3766×10^{-4}	1.2874×10^{-3}	5.1326×10^{-1}	1.2322×10^{-1}	1.8048×10^{-1}

Table 2. Absolute RMS error for the solution of the 2D cantilever beam problem (Second case)

N	**c**	ϵ_{u_1}	ϵ_{u_2}	$\epsilon_{\sigma_{11}}$	$\epsilon_{\sigma_{12}}$	$\epsilon_{\sigma_{22}}$
225	0.5	9.2325×10^{-5}	5.6874×10^{-5}	2.3068×10^{-1}	1.1179×10^{-1}	1.3101×10^{-1}
477	0.5	9.0724×10^{-5}	3.7754×10^{-5}	1.5444×10^{-1}	9.1667×10^{-2}	1.0034×10^{-1}
632	0.3	2.0662×10^{-5}	1.3341×10^{-5}	6.3348×10^{-2}	2.0396×10^{-2}	3.3162×10^{-2}
872	0.1	1.0152×10^{-5}	7.0452×10^{-6}	2.3775×10^{-2}	1.4792×10^{-2}	1.7848×10^{-2}

Table 3. Absolute RMS error for the solution of the plate with a hole problem

Figure 1. Nodal distribution in the solution domain: internal (•) and boundary (◦) points

Figure 2. Geometrical description of the 2-D cantilever beam problem

Figure 3. σ_{11} control plot; analytical (a) and numerical (b) results

Figure 4. Variation of the longitudinal displacement u_1 absolute RMS error in terms of the shape parameter.

Figure 5. Geometrical description of the infinite plate with a hole

Figure 6. Nodal distributions for the plate with a hole problem: a. $N = 225$, b. $N = 477$, c. $N = 632$, d. $N = 872$

Figure 7. Comparison between analytical (top) and numerical (bottom) stress contours: a. σ_{11} b. σ_{12} c. σ_{22}

Figure 8. RMS error in terms of the shape parameter for: a. displacements, b. stresses